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# SPHERE THEOREMS VIA ALEXANDROV FOR CONSTANT WEINGARTEN CURVATURE HYPERSURFACES—APPENDIX TO A NOTE OF A. ROS

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In the paper *Compact hypersurfaces with constant scalar curvature and a congruence theorem*, A. Ros proves that compact embedded hypersurfaces of $\mathbf{R}^{n+1}$ without boundary and with constant scalar curvature must be Euclidean spheres [8]. He uses an integral technique originated by R. C. Reilly for the corresponding mean curvature problem [7], rather than the classical reflection method due to A. D. Alexandrov [1]. His result requires no ellipticity assumption on the surface (e.g. convexity of the compact region it bounds), unlike previous proofs (see [1], or e.g. [6], [9], and their references).

In analogy to an early step of Ros' proof (where it is shown that constant scalar curvature implies positive mean curvature on any candidate surface), we note here that any candidate surface with constant intermediate curvature $H_r$ (the $r$th symmetric function of the $n$ principal curvatures, $1 \leqslant r \leqslant n$) is automatically elliptic. This point is also almost made in Remark 5.B of R. Walter's work [9]. It is a simple yet surprising observation, letting Alexandrov reflection be applied in full generality. In the spirit of L. Caffarelli, L. Nirenberg, and J. Spruck [2], [3], we have the result:

**Theorem.** *Let* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, *with* $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. *Let* $f = f(\lambda)$ *be a function of* $\lambda$ *which satisfies the following conditions*:

(i) $f(\lambda) > 0$ *whenever* $\lambda > \mathbf{0}$ (*each* $\lambda_i > 0$).

(ii) *On the component* $\Gamma$ *of* $\{\lambda \mid f(\lambda) > 0\}$ *containing all positive* $\lambda$, $f$ *is concave* (*i.e., negative Hessian*).

*Then any compact, embedded hypersurface* $\mathbf{S}$ *of* $\mathbf{R}^{n+1}$ *without boundary on which each principal curvature vector* $\lambda = \lambda(\mathbf{P})$, $\mathbf{P} \in \mathbf{S}$, *has the same value* $f(\lambda)$, *is a Euclidean sphere.*

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**Remark 1.** In [4] it is shown that $f = (H_r)^{1/r}$ satisfies (ii) for $2 \leqslant r \leqslant n$. Also, one can show that in this case $\Gamma$ is the set of $\lambda$ for which $H_r$ and all partials (to order $r$) are positive.

**Remark 2.** Condition (ii) implies that $\Gamma$ is convex as well as the ellipticity of $f$ on $\Gamma$ [3]. (If $\lambda$ is the eigenvalue list for a symmetric matric $\mathbf{A}$ and if $f(\lambda) = F(\mathbf{A})$, then ellipticity can be expressed as $\partial f/\partial\lambda_i > 0$ $\forall i$, or by $[\partial F/\partial A_{ij}] > 0$.)

*Proof of the Theorem.* By comparison with a hypersphere, all principal curvatures of $\mathbf{S}$ at an extreme point $\mathbf{P}$ (one at maximum distance from the origin) are at least $1/|\mathbf{P}|$. Thus by (i), (ii) the constant value of $f$ is positive on $\mathbf{S}$ and $\lambda(\mathbf{P}) \in \Gamma$. But because $\mathbf{S}$ is connected and (the ordered) $\lambda$ varies continuously on $\mathbf{S}$, (ii) implies that $\lambda(\mathbf{Q}) \in \Gamma\ \forall \mathbf{Q} \in \mathbf{S}$. Thus by Remark 2 all of $\mathbf{S}$ is elliptic. The ellipticity of $\mathbf{S}$ and the convexity of $\Gamma$ imply that Alexandrov's reflection method will work (Theorem A of [1]). Our generality is a slight extension of Alexandrov's, since his $\lambda$ are positive. Thus we show here that the essential details carry through.

In Alexandrov reflection one can express $\mathbf{S}$ and its reflection locally (near a point of tangency) as graphs above their common tangent plane, where both functions satisfy a fully nonlinear equation:

$$(1) \qquad G(\mathbf{D}u, \mathbf{D}^2 u) = c > 0, \qquad G(\mathbf{0}, \mathbf{A}) = F(\mathbf{A}) = f(\lambda).$$

One wants to show that there cannot be two distinct solutions $u, v$ to (1) satisfying:

$$(2) \quad u \geqslant v \text{ in } \Omega, \quad \mathbf{0} \in \partial\Omega, \quad \partial\Omega \in C^2, \quad u(\mathbf{0}) = 0, \quad \mathbf{D}u(\mathbf{0}) = \mathbf{D}v(\mathbf{0}) = 0.$$

But for $w = u - v$ the mean value theorem implies that the following pointwise equation holds:

$$(3) \qquad 0 = G(\mathbf{D}u, \mathbf{D}^2 u) - G(\mathbf{D}v, \mathbf{D}^2 v) = \frac{\partial G}{\partial u_k}(\mathbf{p}, \mathbf{r}) w_k + \frac{\partial G}{\partial u_{ij}}(\mathbf{p}, \mathbf{r}) w_{ij},$$

$$(\mathbf{p}, \mathbf{r}) = s(\mathbf{D}u, \mathbf{D}^2 u) + (1 - s)(\mathbf{D}v, \mathbf{D}^2 v), \quad \text{some } 0 \leqslant s \leqslant 1.$$

Remark 2, (1) and $D^2u(0) \geqslant D^2v(0)$ imply that $D^2u(0) = D^2v(0)$. Because $\mathbf{D}u$ and $\mathbf{D}v$ are small near the origin, and $\mathbf{D}^2u$ and $\mathbf{D}^2v$ are nearly $D^2u(0)$, the ellipticity of $F$, (1) and the smoothness of $G$ imply that equation (3) is uniformly elliptic near the origin. One can now follow the proof of the Boundary Point Lemma in [5], using only the uniform ellipticity and boundedness of coefficients in (3), to show that $w$ is identically 0.

Because the only tool used in Alexandrov reflection is the boundary point lemma above (and its interior consequence), we conclude that the desired theorem is true.

**Remark 3.** As Alexandrov notes in [1, e], for the purposes of his sphere theorems, reflection in hyperbolic space $\mathbf{H}^{n+1}$ or the upper hemisphere of $\mathbf{S}^{n+1}$ are essentially the same as reflection in $\mathbf{R}^{n+1}$. This theorem and proof remain true in those settings.

**Added in proof.** Condition (ii) in the sphere theorem can be replaced by the more general assumption that $f$ be elliptic on the set $\Gamma$ $(\partial f/\partial\lambda_i > 0\ \forall i)$, and the same proof remains valid.

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